An Exact Solution for the Free Vibration Analysis of Timoshenko Beams

Ramazan A. Jafari-Talookolaei[*1], Maryam Abedi[2]

School of Mechanical Engineering, Babol Noshirvani University of Technology, 47148 – 71167, Babol, Mazandaran Province, Iran

[*1]ramazanali@gmail.com, [2]maryamabedy2000@yahoo.com

Received 27 May 2013; Accepted 18 June 2013; Published 13 March 2014

Abstract

This work presents a new approach to find the exact solutions for the free vibration analysis of a beam based on the Timoshenko type with different boundary conditions. The solutions are obtained by the method of Lagrange multipliers in which the free vibration problem is posed as a constrained variational problem. The Legendre orthogonal polynomials are used as the beam eigenfunctions. Natural frequencies and mode shapes of various Timoshenko beams are presented to demonstrate the efficiency of the methodology.

Keywords

Timoshenko Beam; Natural Frequencies; Mode Shapes; Legendre Polynomials; Lagrange Multipliers

Introduction

Beams play an important role in the creation of mechanical, electromechanical, and civil systems. Many of these systems are subjected to dynamic excitation. As a consequence, the exact determination of the natural frequencies and mode shapes of linear elastic beams have been studied by many researchers. It has been known for many years that the classical Euler-Bernoulli beam theory is able to predict the frequencies of flexural vibration of the lower modes of thin beams with adequate precision. The vibratory motion of thick beams is described by the Timoshenko beam theory, as they incorporate the effects of rotary inertia and deformation due to shear. During the past decades, the free vibrations of Euler–Bernoulli beams have received considerable attention of many researchers, but only few publications were devoted to including the effects of shear deformation and rotary inertia.

The mode shape differential equation describing the transverse vibrations of a hanging Euler–Bernoulli beam under linearly varying axial force has been derived by Schafer (1985). Lee and Ng (1994) have computed the fundamental frequencies and the critical buckling loads of simply supported beams with stepped variation in thickness using two algorithms based on the Rayleigh-Ritz method. The first algorithm which has been used extensively in analyzing beams with non-uniform thickness, involves using a series of assumed functions that satisfy only the external boundary conditions and disregard the presence of the step. The second algorithm considers a beam with a step as two separate beams divided by the step. Two different sets of admissible functions which satisfy the respective geometric boundary conditions have been assumed for these two fictitious sub-beams. Geometric continuities at the step have been enforced by introducing artificial linear and torsional springs.

Lee and Kes (1990) have conducted a study to determine the natural frequencies of non-uniform Euler beams resting on a non-uniform foundation with general elastic end restraints. The free vibration response of an Euler-Bernoulli beam supported by an intermediate elastic constraint has been studied by Riedel and Tan (1998) using the transfer function method. Rosa and Maurizi (1998) have investigated the influence of concentrated masses and Pasternak soil on free vibration of beams and gave exact solutions for Bernoulli–Euler beams based on the beam theory. A modified finite difference method has been presented by Chen and Zhao (2005) to simulate transverse vibrations of an axially moving string.

Lin and Tsai (2007) have determined the natural frequencies and mode shapes of Bernoulli–Euler

multi-span beam carrying multiple spring-mass systems. An analytical solution has been presented for the natural frequencies, mode shapes and orthogonality conditions of an arbitrary system of Euler–Bernoulli beams interconnected by arbitrary joints and subject to arbitrary boundary conditions by Wiedemann (2007). Failla and Santini (2008) have addressed the eigenvalue problem of the Euler–Bernoulli discontinuous beams. A simulation method called the differential transform method (DTM) has been employed to predict the vibration of an Euler–Bernoulli beam (pipeline) resting on an elastic soil by Balkaya and Kaya (2009). Alim and Akkurt (2011) have investigated the free vibration analysis of straight and circular beams on elastic foundation based on the Timoshenko beam theory. Ordinary differential equations in scalar form obtained in the Laplace domain are solved numerically using the complementary functions method.

He and Huang (1987) have used the dynamic stiffness method to analyze the free vibration of continuous Timoshenko beam. The full development and analysis of four models for the transversely vibrating uniform beam have been presented by Han et al. (1999). The four theories namely the Euler-Bernoulli, Rayleigh, shear and Timoshenko have been considered. Zhou (2001) has studied the free vibration of multi-span Timoshenko beams by the Rayleigh-Ritz method. The static Timoshenko beam functions have been developed as the trial functions in the analysis which are the complete solutions of transverse deflections and rotational angles of the beam when a series of static sinusoidal loads acts on the beam.

A study of the free vibration of Timoshenko beams has been presented by Lee and Schultz (2004) on the basis of the Chebyshev pseudospectral method. Chen et al. (2004) have proposed a mixed method, which combines the state space method and the differential quadrature method, for bending and free vibration of arbitrarily thick beams resting on a Pasternak elastic foundation. The Laplace transform has been used to obtain a solution for a Timoshenko beam on an elastic foundation with several combinations of discrete in-span attachments and with several combinations of attachments at the boundaries by Magrab (2007).

In the present paper, a novel approach is made to the problem of the free vibrations of a Timoshenko beam, in which the orthogonal Legendre polynomials in conjunction with Lagrange multipliers are used. The frequencies and the corresponding mode shapes for common types of boundary conditions are compared extremely well with the available solution.

Problem Formulation

Consider a straight Timoshenko beam of length L, a uniform cross-sectional area $A(=b\times h)$, the mass per unit length of m, the second moment of area of the cross-section I, Young's modulus E, and shear modulus G. It is assumed that the beam is made of a homogenous and isotropic material.

The kinetic energy T and the strain energy U of the vibrating beam can be written as:

$$T=\frac{1}{2}\int_0^L\left\{mw_{,t}{}^2+mr^2\psi_{,t}{}^2\right\}d\hat{x} \qquad (1)$$

$$U=\frac{1}{2}\int_o^L\left\{M\psi_{,\hat{x}}+V\gamma\right\}d\hat{x}=\frac{1}{2}\int_0^L\left\{EI\psi_{,x}{}^2+kAG\gamma^2\right\}dx \qquad (2)$$

Where γ represents the shear angle ($\gamma=w_{,\hat{x}}-\psi$), $w(\hat{x},t)$ and $\psi(\hat{x},t)$ are transverse displacement and the cross-section rotation due to the bending moment, $\hat{x}$ is the axial coordinate of the beam, r is the radius of gyration ($=\sqrt{I/A}$) and k is the beam cross sectional shape factor. Also M and V are the bending moment and shear forces, respectively. Comma denotes differentiation with respect to $\hat{x}$ or t.

Applying Hamilton's principle, the governing equations of motion and boundary conditions are obtained as follows:

$$mw_{,tt}-(kAG(w_{,\hat{x}}-\psi))_{,x}=0 \qquad (3)$$

$$mr^2\psi_{,tt}-kAG(w_{,\hat{x}}-\psi)-(EI\psi_{,x})_{,x}=0 \qquad (4)$$

$$(M\delta\psi)_0^L=0,\ (V\delta w)_0^L=0 \qquad (5)$$

in other words, at the ends $\hat{x}$=0 and L, we have:

$$\begin{cases}\text{either } M=0 \text{ or } \psi=0 \text{ is specified,}\\ \text{either } V=0 \text{ or } w=0 \text{ is specified.}\end{cases} \qquad (6)$$

The equation (6) gives the boundary conditions of the present case.

Analytical Solution

In the present work, series of solutions in conjunction with the Lagrange multipliers are used to study the free vibration characteristics of the beam. The main advantage of the Lagrange multiplier technique is that the choice of the assumed displacement function is easy because they do not have to satisfy the boundary conditions of the problem. In the present study, the simple Legendre polynomials are chosen as displacement functions, and this simplifies the problem further since the orthogonality properties

lead to simple energy expression.

Harmonic solutions for the variables $w(\hat{x},t)$ and $\psi(\hat{x},t)$ are assumed as:

$$w(\hat{x},\hat{t}) = W(x)e^{i\omega t}, \quad \psi(x,t) = \Psi(x)e^{i\omega t} \quad (7)$$

in which variables $W(\hat{x})$ and $\Psi(\hat{x})$ are the displacement functions and ω is the circular frequency. As mentioned above, the displacement functions can be expressed in terms of the simple Legendre polynomials and are given by:

$$W(x) = \sum_{m=0}^{n_t} W_m P_m(x), \quad \Psi(x) = \sum_{m=0}^{n_t} \Psi_m P_m(x) \quad (8)$$

Here, Pm is the simple Legendre polynomial of degree m. It should be mentioned that the axial coordinate is transformed to the interval $-1 \le x \le 1$ by letting $x = \frac{\hat{x} - L/2}{L/2}$.

We have four boundary conditions for each beam, i.e. two boundary conditions for two ends. These four boundary conditions which are not satisfied by the assumed series, are imposed as constraints. For four common boundary conditions (B.C.s), these constraints can be written as follows:

Clamped-Clamped Beam (C-C):

$$\begin{aligned} W(-1) = 0, \quad \Psi(-1) = 0 \\ W(1) = 0, \quad \Psi(1) = 0 \end{aligned} \quad (9a)$$

Clamped-Hinged Beam (C-H):

$$\begin{aligned} W(-1) = 0, \quad \Psi(-1) = 0 \\ W(1) = 0, \quad \Psi'(1) = 0 \end{aligned} \quad (9b)$$

Hinged-Hinged Beam (H-H):

$$\begin{aligned} W(-1) = 0, \quad \Psi'(-1) = 0 \\ W(1) = 0, \quad \Psi'(1) = 0 \end{aligned} \quad (9c)$$

Clamped- Free Beam (C-F):

$$\begin{aligned} W(-1) = 0, \quad \Psi(-1) = 0 \\ \frac{2}{L}W'(1) - \Psi(1) = 0, \quad \Psi'(1) = 0 \end{aligned} \quad (9d)$$

in which prime denotes differentiation with respect to x. The boundary conditions yield linear constraints related to the linear combinations of the Legendre polynomials and subject to the degree of approximation, i.e. the number of polynomials involved. By substituting equations (8) in equation (9), the constraints can be rewritten as:

Clamped-Clamped Beam (C-C):

$$\begin{aligned} \sum_{m=0}^{n_t} (-1)^m W_m = 0, \quad \sum_{m=0}^{n_t} (-1)^m \Psi_m = 0, \\ \sum_{m=0}^{n_t} W_m = 0, \quad \sum_{m=0}^{n_t} \Psi_m = 0 \end{aligned} \quad (10a)$$

Clamped-Hinged Beam (C-H):

$$\begin{aligned} \sum_{m=0}^{n_t} (-1)^m W_m = 0, \quad \sum_{m=0}^{n_t} (-1)^m \Psi_m = 0 \\ \sum_{m=0}^{n_t} W_m = 0, \quad \sum_{m=1}^{n_t} \Psi_m \sum_{k_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m - 4k_1 - 1) = 0 \end{aligned} \quad (10b)$$

Hinged-Hinged Beam (H-H):

$$\begin{aligned} \sum_{m=0}^{n_t} (-1)^m W_m = 0, \sum_{m=1}^{n_t} \Psi_m \sum_{k_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-1)^{m-2k_1-1} (2m - 4k_1 - 1) = 0 \\ \sum_{m=0}^{n_t} W_m = 0, \quad \sum_{m=1}^{n_t} \Psi_m \sum_{k_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m - 4k_1 - 1) = 0 \end{aligned} \quad (10c)$$

Clamped- Free Beam (C-F):

$$\begin{aligned} \sum_{m=0}^{n_t} (-1)^m W_m = 0, \quad \sum_{m=0}^{n_t} (-1)^m \Psi_m = 0 \\ \frac{2}{L} \sum_{m=1}^{n_t} W_m \sum_{k_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m - 4k_1 - 1) - \sum_{m=0}^{n_t} \Psi_m = 0 \\ \sum_{m=1}^{n_t} \Psi_m \sum_{k_1=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} (2m - 4k_1 - 1) = 0 \end{aligned} \quad (10d)$$

In above equation, we have used the following properties of Legendre polynomial (Gradshteyn and Ryzhik, 2007):

$$P'_n(x) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (2n - 4k - 1) P_{n-2k-1}(x) \qquad (n \ge 1)$$

in which $\left\lfloor \frac{n-1}{2} \right\rfloor$ signifies the integral part of *(n-1)/2* (Gradshteyn and Ryzhik, 2007).

A variational principle is formulated based on the kinetic and strain energies by a procedure similar to one followed by Washizu (1982). This variational principle along with the constraint conditions is used to solve the vibration problem. The function to be extremized is given by the expression:

$$F = U + T - \sum_{i=1}^{4} \alpha_i (\text{constraints equation}) \quad (11)$$

where α_i $(i = 1-4)$ are the Lagrange multipliers. Substituting the assumed series for $W(x)$ and $\Psi(x)$ in equation (11) and simplifying yields:

$$F=\frac{EI}{L}\int_{-1}^{1}\sum_{m=1}^{n_t}\Psi_m\sum_{k_1=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(2m-4k_1-1)P_{m-2k_1-1}(x)\sum_{n=1}^{n_t}\Psi_n\sum_{k_2=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_2-1)P_{n-2k_2-1}(x)\,dx$$
$$+\frac{kAGL}{4}\sum_{m=0}^{n_t}\frac{2}{2m+1}\Psi_m^{\,2}$$
$$+\frac{kAG}{L}\int_{-1}^{1}\sum_{m=1}^{n_t}W_m\sum_{k_1=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(2m-4k_1-1)P_{m-2k_1-1}(x)\sum_{n=1}^{n_t}W_n\sum_{k_2=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}(2n-4k_2-1)P_{n-2k_2-1}(x)\,dx$$
$$-kAG\int_{-1}^{1}\sum_{m=0}^{n_t}\Psi_m P_m(x)\sum_{n=1}^{n_t}W_n\sum_{k_1=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(2n-4k_1-1)P_{n-2k_1-1}(x)\,dx$$
$$-\frac{m\omega^2L}{4}\sum_{m=0}^{n_t}\frac{2}{2m+1}\left(W_m^{\,2}+r^2\Psi_m^{\,2}\right)-\sum_{i=1}^{4}\alpha_i(\text{constraints equation}) \tag{12}$$

The necessary extremizing conditions are given by:

$$\frac{\partial F}{\partial W_m}=\frac{\partial F}{\partial \Psi_m}=0,\quad m=0,1,2,... \tag{13}$$

Using equation (13) in conjunction with equation (12) results in a system of linear algebraic equations which, in matrix form, can be written as:

$$\underline{\underline{A}}\left\{W_0,W_1,...,W_{n_t},\Psi_0,\Psi_1,...,\Psi_{n_t}\right\}^T=\underline{\underline{B}} \tag{14}$$

in which the right hand side of equation (14) consists of Lagrange multipliers. Solving equation (14) for W_m and Ψ_m $(m=1,2,...,n_t)$ and substituting into the constraint equations (10) results in a system of homogenous linear algebraic equations with the Lagrange multipliers as unknowns. The system of equations is given by:

$$\underline{\underline{C}}\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^T=\underline{0} \tag{15}$$

The natural frequencies and corresponding mode shapes of beams can be calculated using equations (14) and (15). In calculating the natural frequency, the determinant of the coefficient matrix in equation (15) is computed for various values of frequency starting from a near zero value. Determinant change of sign function is identified and the corresponding value of frequency is the natural frequency of the beam in question.

Results and Discussion

In order to demonstrate the high accuracy of the present method, the convergence and comparison studies are carried out. Unless mentioned otherwise, in all of the following analysis the rectangular cross-sectional beams with shear correction factor k=5/6 and the Poisson ratio υ=0.3 are considered. The first five dimensionless frequencies ($\Omega=\sqrt{\frac{mL^4}{EI\pi^4}}\omega$) of hinged-hinged (H-H) and clamped-clamped (C-C) beams are given in Table 1. The number of terms of the Legendre polynomial steadily increases from 6 to 10. One can see that the convergence is very rapid. In general, 10 terms of the Legendre polynomial are enough to give satisfactory results.

TABLE 1 THE CONVERGENCE STUDY ON THE FIRST FIVE DIMENSIONLESS FREQUENCIES OF H-H AND C-C TIMOSHENKO BEAMS FOR (L/H=10)

B.C.	n_t	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
H-H	6	2.1251	5.4903	10.1815	21.8025	34.7663
	7	2.1251	5.4467	10.0291	15.8141	30.3098
	8	2.1251	5.4461	9.8518	15.4366	22.2419
	9	2.1251	5.4456	9.8425	14.9628	20.6510
	10	2.1251	5.4456	9.8425	14.9628	20.6507
C-C	6	0.9836	3.7670	8.1029	17.3346	37.0371
	7	0.9836	3.7588	7.9795	13.6489	26.0859
	8	0.9836	3.7588	7.9211	13.2646	20.2042
	9	0.9836	3.7588	7.9189	13.0420	18.8547
	10	0.9836	3.7588	7.9189	13.0419	18.8543

The comparison study has been given in Table 2 for the first five dimensionless frequencies of Timoshenko beams by using the present method, the dynamic stiffness method (DSM) (He and Huang, 1987) and static Timoshenko beam function (STBF) (Lee and Schultz, 2004). Three types of boundary conditions: H-H, C-H and CC have been considered. Excellent agreement has been observed for all cases, which shows that the present method has very high accuracy.

TABLE 2 THE COMPARISON STUDY OF THE FIRST FIVE DIMENSIONLESS FREQUENCIES OF H-H, C-H AND C-C TIMOSHENKO BEAMS FOR (L/H=6.667)

B.C.	Methods	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
H-H	Present	0.9644	3.5195	7.0424	11.0702	15.3445
	DSM	0.9644	3.5194	7.0424	11.0702	15.3444
	STBF	0.9644	3.5194	7.0424	11.0702	15.3444
C-H	Present	1.4387	4.1632	7.6625	11.5809	15.7282
	DSM	1.4386	4.1632	7.6625	11.5807	15.7273
	STBF	1.4386	4.1633	7.6626	11.5814	15.7293
C-C	Present	1.9814	4.7859	8.2462	12.0601	16.0888
	DSM	1.9814	4.7859	8.2461	12.0580	16.0887
	STBF	1.9814	4.7860	8.2462	12.0604	16.0889

Fig. 1 shows the effect of length-to-thickness ratio (L/h), where L varies while h keeps constant, on the dimensionless fundamental frequency of the beam with C-C, C–H, H-H and C-F boundary conditions. It

is clear that for the beam with L/h<30, the variation of L/h has drastic effect on Ω. While for higher values of L/h, the fundamental frequency tends to be constant, that is, the influence of L/h is practically negligible. As expected, the curve shows clearly that the smaller the length-to-thickness ratio is, the lower the frequency will be.

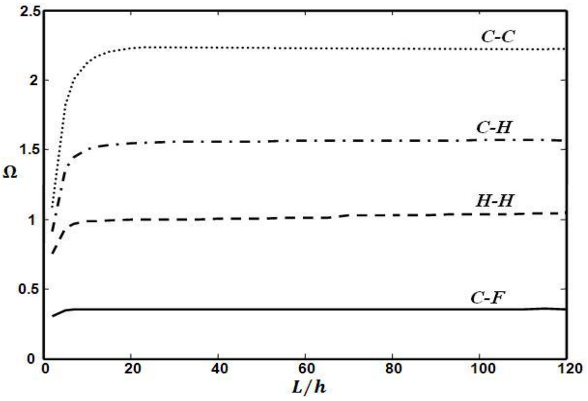
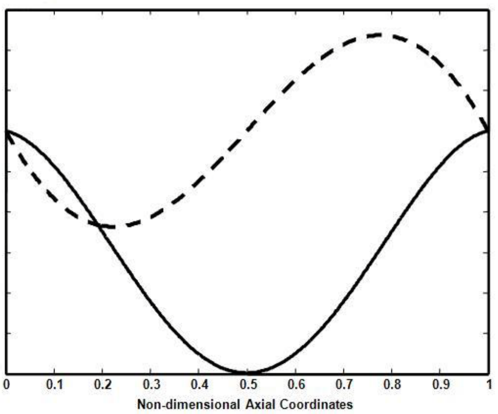


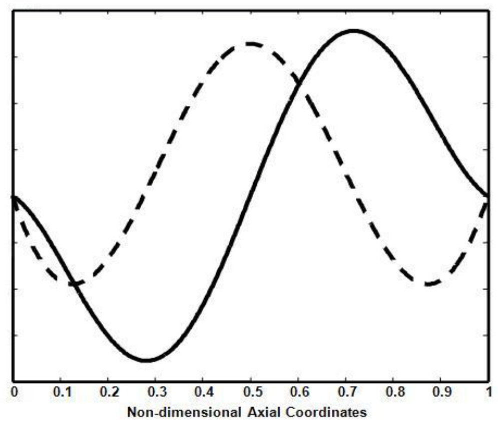
FIG. 1 EFFECT OF LENGTH-TO-THICKNESS RATIO ON THE DIMENSIONLESS FUNDAMENTAL NATURAL FREQUENCIES OF THE BEAM WITH VARIOUS BOUNDARY CONDITIONS

Fig. 2 shows the variation of the displacement Wand the bending slope Ψ along the beam length for the first three modes of vibration of Timoshenko beam with thickness ratio L/h=10 and C-C boundary condition.

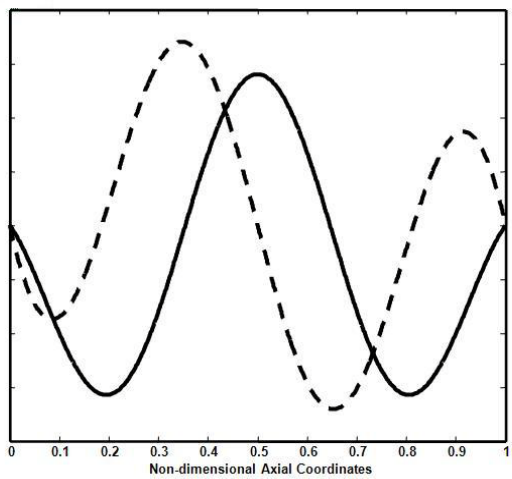
Since the conventional beam theories can not involve the effect of Poisson's ratio, it is rather interesting to take a deep insight into it using the present approach. Table 3 gives the variation of the first three natural frequency parameters (Ω) of H-H beams with the Poisson's ratio. It is shown that the natural frequency decreases gradually with the increasing of Poisson's ratio. We can see that the natural frequencies for υ=0.5 have an apparent deviation from that for υ=0.1. From this point of view, the Poisson's ratio is of great significance in structural design especially for composite material beams.



MODE 1



MODE 2



MODE 3

FIG. 2 VARIATION OF DISPLACEMENT AND THE BENDING SLOPE ALONG THE BEAM LENGTH (MODES 1-3)

W (——), Ψ(- - - -)

TABLE 3 EFFECT OF POISSON'S RATIOS ON THE FIRST THREE NATURAL FREQUENCY PARAMETERS (Ω) OF H-H BEAMS

L/h		**Poisson's ratio (υ)**				
		0.1	0.2	0.3	0.4	0.5
10	Ω_1	0.9854	0.9845	0.9836	0.9827	0.9817
	Ω_2	2.1948	2.1897	2.1821	2.1870	2.1818
	Ω_3	3.7844	3.7714	3.7586	3.7459	3.7334
25	Ω_1	0.9978	0.9977	0.9975	0.9974	0.9972
	Ω_2	3.9625	3.9600	3.9576	3.9551	3.9526
	Ω_3	8.8136	8.8017	8.7900	8.7782	8.7665

Conclusions

The free vibration of the Timoshenko beams is investigated using an assumed series solution in conjunction with Lagrange multipliers. It is observed that the present method is a computationally efficient

tool in predicting the natural frequencies of the beams. This method is particularly attractive because of the ease with which one can choose the generalized displacement functions. This fact is shown by choosing Legendre Polynomials whose orthogonal properties simplify energy expression considerably. The natural frequencies of the Timoshenko beam obtained by this method compare extremely well with the available exact solution. It should be mentioned that by applying this method, the convergence is very rapid.

REFERENCES

Alim, F.F., Akkurt, F.G., "Static and free vibration analysis of straight and circular beams on elastic foundation." Mechanics Research Communications 38 (2011): 89-94.

Balkaya Müge, Kaya Metin O. and Saˇglamer Ahmet, "Analysis of the vibration of an elastic beam supported on elastic soil using the differential transform method." Archive of Applied Mechanics 79(2009): 135–46.

Chen Li-Qun, and Zhao Wei-Jia, "A numerical method for simulating transverse vibrations of an axially moving string." Applied Mathematics and Computation 160 (2005): 411–22.

Chen W.Q., Lu C.F., Bian Z.G., "A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation." Applied Mathematical Modelling 28 (2004): 877–90.

Failla Giuseppe and Santini Adolfo, "A solution method for Euler–Bernoulli vibrating discontinuous beams." Mechanics Research Communications 35 (2008): 517–29.

Gradshteyn I.S. and Ryzhik I.M., Table of Integrals, Series, and Products, Seventh Edition, Elsevier Inc. 2007.

Han Seon M., Benaroya Haym and Wei Timothy, "Dynamics of Transversely Vibrating Beams Using Four Engineering Theories." Journal of Sound and Vibration 225 (1999): 935-88.

He Y.-S. and Huang T. C., 1987 Advanced Topics in Vibrations: presented at American Society of Mechanical Engineers Design Technology Conferences-11th Biennial Conference on Mechanical Vibration and Noise, 43-48, New York. Free Vibration analysis of continuous Timoshenko beams by dynamic stiffness method.

Lee H. P., and Ng T. Y., "Vibration and Buckling of a Stepped Beam." Applied Acoustics 42 (1994) 257-66.

Lee J. and Schultz W.W., "Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method." Journal of Sound and Vibration 269 (2004): 609–21.

Lee S.Y., and Kes H.Y., "Free vibrations of non-uniform beams resting on non-uniform elastic foundation with general elastic end restraints." Comput. Struct. 34 (1990): 421–29.

Lin HY and Tsai YC, "Free vibration analysis of a uniform multi-span beam carrying multiple spring-mass systems." Journal of Sound and Vibration 302 (2007): 442–56.

Magrab Edward B., "Natural Frequencies and Mode Shapes of Timoshenko Beams with Attachments." Journal of Vibration and Control, 13(2007): 905-34.

Riedel C.H., and Tan C.A., "Dynamic Characteristics and Mode Localization of Elastically Constrained Axially Moving Strings and Beams." Journal of Sound and Vibration 215 (1998): 455-73.

Rosa M.A. De, and Maurizi M.J., "The influence of concentrated masses and Pasternak soil on the free vibrations of Euler beams-exact solution." Journal of Sound and Vibration 212 (1998): 573–81.

Schafer B., "Free vibrations of a gravity-loaded clamped-free beam." Ingenieur-Archiv 55 (1985): 66–80.

Washizu, K. 1982, Variational Methods in Elasticity and Plasticity, New York: Pergamon Press.

Wiedemann S.M., "Natural frequencies and mode shapes of arbitrary beam structures with arbitrary boundary conditions." Journal of Sound and Vibration 300 (2007): 280–91.

Zhou D., "Free Vibration of Multi-Span Timoshenko Beams Using Static Timoshenko Beam Functions." Journal of Sound and Vibration 241 (2001): 725-34.